Real interpolation theory for Herz spaces

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In this short note, we show the characterization of real interpolation spaces of Herz spaces:

$$\|f\|_{\dot{K}^{\alpha}_{p,q}} := \left\| \left\{ 2^{j\alpha} \|\chi_j f\|_{L^p} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q}.$$

This characterization was already proved by Gilbert [1]. But, we give a proof.

Theorem 0.1. *Let $0 < p, q \le \infty$ and $-\infty < \alpha_0 < \alpha < \alpha_1 < \infty$. For $\theta \in (0,1)$ with $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, it follows that*

$$\dot{K}^{\alpha}_{p,q} = \left(\dot{K}^{\alpha_0}_{p,1}, \dot{K}^{\alpha_1}_{p,1}\right)_{\theta,q}.$$

Proof. Let $\tau := -(\alpha-\alpha_0)/\theta < 0$.

$$\begin{aligned}
\|f\|_{(\dot{K}^{\alpha_0}_{p,1},\dot{K}^{\alpha_1}_{p,1})_{\theta,q}} &= \left(\int_0^\infty \left(\lambda^{-\theta} K(f,\lambda)\right)^q \frac{d\lambda}{\lambda} \right)^{1/q} \\
&\approx \left(\sum_{j\in\mathbb{Z}} 2^{-j\theta\tau q} K(f, 2^{j\tau q})^q \right)^{1/q} \\
&\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{-j\theta\tau q} \left\| \sum_{k\ge j} \chi_k f \right\|^q_{\dot{K}^{\alpha_0}_{p,1}} \right)^{1/q} + \left(\sum_{j\in\mathbb{Z}} 2^{j(1-\theta)\tau q} \left\| \sum_{k\le j} \chi_k f \right\|^q_{\dot{K}^{\alpha_1}_{p,1}} \right)^{1/q} \\
&\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{-j\theta\tau q} \left(\sum_{\ell\ge j} 2^{\ell\alpha_0} \|\chi_\ell f\|_{L^p} \right)^q \right)^{1/q} + \left(\sum_{j\in\mathbb{Z}} 2^{j(1-\theta)\tau q} \left(\sum_{\ell\le j} 2^{\ell\alpha_1} \|\chi_\ell f\|_{L^p} \right)^q \right)^{1/q} \\
&:= I + II.
\end{aligned}$$

In the case, $q \in (0,1]$,

$$\begin{aligned}
I &\le \left(\sum_{j\in\mathbb{Z}} 2^{-j\theta\tau q} \sum_{\ell\ge j} 2^{\ell\alpha_0 q} \|\chi_\ell f\|^q_{L^p} \right)^{1/q} \\
&= \left(\sum_{\ell\in\mathbb{Z}} 2^{\ell\alpha_0 q} \|\chi_\ell f\|^q_{L^p} \sum_{j\le \ell} 2^{-j\theta\tau q} \right)^{1/q} \\
&\lesssim \|f\|_{\dot{K}^{\alpha}_{p,1}}.
\end{aligned}$$

In the case $q = \infty$,

$$I = \sup_{j\in\mathbb{Z}} 2^{-j\theta\tau} \sum_{\ell\ge j} 2^{\ell\alpha_0} \|\chi_\ell f\|_{L^p} \lesssim \|f\|_{\dot{K}^{\alpha}_{p,\infty}}.$$

To see the case $q \in (1,\infty)$, we consider the operator

$$T : \left\{2^{j\alpha}\|\chi_j f\|_{L^p}\right\}_{j\in\mathbb{Z}} \mapsto \left\{ 2^{-j\theta\tau} \left(\sum_{\ell\ge j} 2^{\ell\alpha_0} \|\chi_\ell f\|_{L^p} \right) \right\}_{j\in\mathbb{Z}} = \left\{ 2^{j(\alpha-\alpha_0)} \left(\sum_{\ell\ge j} 2^{-\ell(\alpha-\alpha_0)} 2^{\ell\alpha} \|\chi_\ell f\|_{L^p} \right) \right\}_{j\in\mathbb{Z}}.$$

We now know that T is bounded on ℓ^1 and ℓ^∞. Interpolating them, one obtains the ℓ^q-boundedness.

The same argument as above gives us that $II \lesssim \|f\|_{\dot{K}^{\alpha}_{p,q}}$. Thus, it holds $\dot{K}^{\alpha}_{p,q} \hookrightarrow \left(\dot{K}^{\alpha_0}_{p,1}, \dot{K}^{\alpha_1}_{p,1}\right)_{\theta,q}$. □

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For the opposite direction,

$$\|f\|_{\dot{K}^{\alpha}_{p,q}} \lesssim \left(\sum_{j\in\mathbb{Z}} 2^{j\alpha q} \|\chi_j f\|^q_{L^p} \right)^{1/q}.$$

For arbitrary decomposition $f = f_0 + f_1$, we can easily see that

$$\begin{aligned} 2^{j\alpha} \|\chi_k f\|_{L^p} &\le 2^{j(\alpha-\alpha_0)} \left(2^{j\alpha_0} \|\chi_j f_0\|_{L^p} + 2^{j(\alpha_0-\alpha_1)} \|\chi_j f_1\|_{L^p} \right) \\ &\le 2^{j(\alpha-\alpha_0)} \left(\|f_0\|_{\dot{K}^{\alpha_0}_{p,\infty}} + 2^{j(\alpha_0-\alpha_1)} \|f_1\|_{\dot{K}^{\alpha_1}_{p,\infty}} \right). \end{aligned}$$

therefore,

$$\begin{aligned} \|f\|_{\dot{K}^{\alpha}_{p,q}} &\le \left(\sum_{j\in\mathbb{Z}} 2^{j(\alpha-\alpha_0)q} K(f, 2^{j(\alpha_0-\alpha_1)}; \dot{K}^{\alpha_0}_{p,\infty}, \dot{K}^{\alpha_1}_{p,\infty})^q \right)^{1/q} \\ &\lesssim \left(\sum_{j\in\mathbb{Z}} \int_{2^{j\tau}}^{2^{(j+1)\tau}} \left(\lambda^{(\alpha-\alpha_0)/\tau} K(f, \lambda^{(\alpha_0-\alpha_1)/\tau}; \dot{K}^{\alpha_0}_{p,\infty}, \dot{K}^{\alpha_1}_{p,\infty})^q) \right)^q \frac{d\lambda}{\lambda} \right)^{1/q} \\ &\lesssim \|f\|_{(\dot{K}^{\alpha_0}_{p,\infty}, \dot{K}^{\alpha_1}_{p,\infty})_{\theta,q}} \\ &\le \|f\|_{(\dot{K}^{\alpha_0}_{p,1}, \dot{K}^{\alpha_1}_{p,1})_{\theta,q}} \end{aligned}$$

Remark 0.1. *The proof above means that for all $r \in [1,\infty]$*

$$\dot{K}^{\alpha}_{p,q} = (\dot{K}^{\alpha_0}_{p,r}, \dot{K}^{\alpha_1}_{p,r})_{\theta,q}.$$

References

G [1] J.E. Gilbert, *Interpolation between of L^p-spaces*, Ark. Mat., **10**, (1972), 235-249.